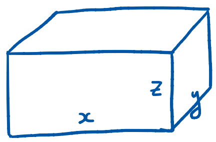
Example 3. A rectangular box is to be made from 100 m^2 of cardboard. Find the maximum volume of such a box.



We want: $\max \overbrace{xyz}^{f(x,y,z)}$

s.t. $\underbrace{\cancel{2}xy + \cancel{2}xz + 2yz}_{g(x,y,z)} = \underbrace{\overset{50}{\cancel{100}}}_{k}$ $\quad (x,y,z>0)$

$\nabla f(x,y,z) = \langle yz,\ xz,\ xy\rangle$ $\qquad \nabla g(x,y,z) = \langle y+z,\ x+z,\ x+y\rangle$

LM equations:

$$yz = \lambda(y+z) \quad ①$$
$$xz = \lambda(x+z) \quad ②$$
$$xy = \lambda(x+y) \quad ③$$
$$xy + xz + yz = 50 \quad ④$$

Note: If $\lambda = 0$, then ①, ②, ③ $\Rightarrow yz = xz = xy = 0$

$\Rightarrow xy + xz + yz = 0$, which contradicts ④

$\Rightarrow \lambda$ must be $\neq 0$.

x times ① $\Rightarrow xyz = \lambda(xy + xz)$ ⑤

y times ② $\Rightarrow xyz = \lambda(xy + yz)$ ⑥

z times ③ $\Rightarrow xyz = \lambda(xz + yz)$ ⑦

b/c $x, y, z > 0$

⑤ + ⑥ $\Rightarrow xy + xz = xy + yz$

$\Rightarrow xz = yz \Rightarrow \cancel{z = 0}$ or $x = y$ ⑧

⑤ + ⑦ $\Rightarrow xy + xz = xz + yz$

$\Rightarrow xy = yz \Rightarrow \cancel{y = 0}$ or $x = z$ ⑨

⑥ + ⑦ $\Rightarrow xy + yz = xz + yz$

$\Rightarrow xy = xz \Rightarrow \cancel{x = 0}$ or $y = z$ ⑩

⑧ + ⑨ + ⑩ $\Rightarrow x = y = z$

b/c $x>0$

Sub into ④ $\Rightarrow x^2 + x^2 + x^2 = 50 \Rightarrow 3x^2 = 50 \Rightarrow x = \sqrt{\frac{50}{3}},\ \cancel{-\sqrt{\frac{50}{3}}}$

Solns to LM eqs: $\left(\sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}\right)$ $\quad f\left(\sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}\right) = \left(\sqrt{\frac{50}{3}}\right)^3$ $\quad$ Abs. min. or max?

Let's try $(1, 1, 25)$, which satisfies $xy + xz + yz = 50$.

$f(1,1,25) = 25 < \left(\sqrt{\frac{50}{3}}\right)^3$

$\Rightarrow f\left(\sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}\right)$ is an absolute maximum, since there is another solution $(1,1,25)$ that satisfies the constraint $xy + xz + yz = 50$ w/ lower f value.